Estimation of Absolute Error for Minimization Maximum Lateness

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In this paper, we consider the approach finding of the approximate decision with the guaranteed absolute error for the problems minimizing maximum lateness. The idea of the approach consists in construction to a initial instance A such instance B (with the same number of jobs) with minimum of estimation of absolute error that

$$0 \leq L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \leq \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B),$$

where

$$\rho_d(A,B) = \max_{j \in N}\{d^A_j - d^B_j\} + \max_{j \in N}\{d^B_j - d^A_j\},$$

$$\rho_r(A,B) = \max_{j \in N}\{r^A_j - r^B_j\} + \max_{j \in N}\{r^B_j - r^A_j\}$$

and

$$\rho_p(A,B) = \sum_{j \in N} |p^A_j - p^B_j|,$$

and π^A, π^B – optimal schedules for instances A and B, respectively. Besides $\rho(A,B) = \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B)$ satisfies to properties of the metrics in $(3n-2)$-dimensional space $\{(r_j, p_j, d_j) \,|\, j \in N\}$ with fixed in two parameters.

Keywords: Algorithmics, Machine Scheduling, Multi-processor Scheduling, Theoretical Scheduling, Estimation Absolute Error.

Introduction

Suppose that m identical machines $M_i, i = 1, \dots, m$, have to process n jobs $J_j, j \in N = \{1, \dots, n\}$. Preemption of the jobs is not allowed. The machines can handle only one job at a time. For each job j, $j \in N$, a release date r_j, a processing time $p_j > 0$ and a due date d_j are given. The precedence relations between jobs may be represented by an acyclic directed graph G.

A schedule π is uniquely determined by a permutation of the elements of N, which consists of m schedules π_i for each machine $M_i, i = 1, \dots, m, \pi = \bigcup_{i=1}^m \pi_i$. The objective function is maximum lateness $L_{max}(\pi) = \max_{j \in N} L_j(\pi)$, where $L_j(\pi) = C_j(\pi) - d_j$, and $C_j(\pi)$ is complete time job $j \in N$ in schedule π.

This problem $P|prec; r_j|L_{max}$ is a generalisation of some unary NP-hard problems, for example: $P|intree; r_j; p_j = 1|C_{max}$, $P|outtree; p_j = 1|L_{max}$ [1]; $P2|chains|C_{max}$ [2]; $P||C_{max}$ [3]; $1|r_j|L_{max}$, $P2||C_{max}$ [4]; $P|prec; p_j = 1|C_{max}$ [5].

For some related problems exist polynomial solvable cases: $P2|prec; r_j; p_j = 1|L_{max}$ [6]; $P|p_j = p; r_j|L_{max}$ [7]; $1|prec; pmtn; r_j|L_{max}$ [8]; $P|chains; r_j; p_j = 1|L_{max}$ [9]; $1|prec; pmtn; r_j|L_{max}$ [10]; $P|chains; r_j; p_j = 1|L_{max}$ [11]; $P2|prec; p_j = p|L_{max}$ [12]; $JMPM|prec; r_j; n = 2|L_{max}$ [13]; $1|prec|L_{max}$ [14]; $1|prec; p_j = p; r_j|L_{max}$ [15].

Estimation of an absolute error for the problem minimizing maximum lateness for single machine $1|r_j|L_{max}$ has been considered in [16, 17].

1 Definitions

Definition 1 *We denote by $L_j^A(\pi)$ and $C_j^A(\pi)$ lateness and complete time of job j in schedule π for instance A with parameters $\{G^A, (r_j^A, p_j^A, d_j^A) | j \in N\}$. And, accordingly, $L_{max}^A(\pi) = \max_{j\in N} L_j^A(\pi)$ and π^A– optimal schedule for instance A.*

Definition 2 *Let's name an instance $B = \{G^B, (r_j^B, p_j^B, d_j^B) \,|\, j \in N\}$* **inverse** *to initial instance $A = \{G^A, (r_j^A, p_j^A, d_j^A) \,|\, j \in N\}$, if*

$$r_j^B = -d_j^A,\ p_j^B = p_j^A,\ d_j^B = -r_j^A,\ \forall\ j \in N.$$

In instance B orientation of all edges of the graph is replaced on opposite, $\overleftarrow{G}^B = \overrightarrow{G}^A$. And schedule $\pi_i' = (j_{n_i}, j_{n_i-1}, \ldots, j_1)$ is **inverse** *to schedule $\pi_i = (j_1, \ldots, j_{n_i-1}, j_{n_i})$ for each machine $i \in M$.*

Definition 3 *For two any instances A and B we'll define following functions:*

$$\rho_d(A,B) = \max_{j\in N}\{d_j^A - d_j^B\} + \max_{j\in N}\{d_j^B - d_j^A\},$$

$$\rho_r(A,B) = \max_{j\in N}\{r_j^A - r_j^B\} + \max_{j\in N}\{r_j^B - r_j^A\},$$

$$\rho_p(A,B) = \sum_{j\in N} |p_j^A - p_j^B|,$$

$$\rho(A,B) = \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B).$$

As for instances $A = \{G, (r_j, p_j, d_j) \,|\, j \in N\}$ and $A' = \{G, (r_j + \alpha, p_j, d_j + \beta) \,|\, j \in N\}$ the set of optimum schedules equals, it is possible "to fix" two parameters, for definiteness, $\alpha = -r_1$ and $\beta = -d_1$. Then the function $\rho(A,B)$ satisfies to properties of normed metrics in $(3n-2)$-dimensional space.

2 Estimation of absolute error

Lemma 1 *Let $A = \{G^A, (r_j, p_j, d_j^A) | j \in N\}$ and $B = \{G^B, (r_j, p_j, d_j^B) | j \in N\}$ (with identical release and processing times $r_j, p_j, j \in N$,) are two instances then for any schedule π holds*

$$L_{max}^B(\pi) - L_{max}^A(\pi) \le \max_{j\in N}\{d_j^A - d_j^B\}. \tag{1}$$

Proof: For any $j \in N$ we have: $L_{max}^A(\pi) + \max_{i\in N}\{d_i^A - d_i^B\} \ge C_j(\pi) - d_j^A + d_j^A - d_j^B = C_j(\pi) - d_j^B$. So, $L_{max}^A(\pi) + \max_{i\in N}\{d_i^A - d_i^B\} \ge \max_{j\in N}\{C_j(\pi) - d_j^B\} = L_{max}^B(\pi)$. □

The instances A and B are "symmetric", so obviously for any schedule π holds

$$L_{max}^A(\pi) - L_{max}^B(\pi) \le \max_{j\in N}\{d_j^B - d_j^A\}. \tag{2}$$

Lemma 2 *Let $A = \{G, (r_j, p_j, d_j^A) | j \in N\}$ and $B = \{G, (r_j, p_j, d_j^B) | j \in N\}$ (with identical release and processing times $r_j, p_j, j \in N$, and graph G) are two any instances then*

$$0 \le L_{max}^A(\pi^B) - L_{max}^A(\pi^A) \le \rho_d(A,B).$$

Proof: From (1), (2) for schedules π^A and π^B we have

$$L^A_{max}(\pi^A) + \max_{j\in N}\{d^A_j - d^B_j\} \geq L^B_{max}(\pi^A), \tag{3}$$

$$L^B_{max}(\pi^B) + \max_{j\in N}\{d^B_j - d^A_j\} \geq L^A_{max}(\pi^B). \tag{4}$$

Schedule π^B is optimal for instance B so

$$L^B_{max}(\pi^A) \geq L^B_{max}(\pi^B). \tag{5}$$

¿From (3)–(5)

$$L^A_{max}(\pi^A) + \max_{j\in N}\{d^A_j - d^B_j\} \geq L^A_{max}(\pi^B) - \max_{j\in N}\{d^B_j - d^A_j\},$$

then

$$L^A_{max}(\pi^A) + \rho_d(A,B) \geq L^A_{max}(\pi^B) \geq L^A_{max}(\pi^A).$$

□

The instances A and B are "symmetric" so obviously $L^B_{max}(\pi^A) - L^B_{max}(\pi^B) \leq \rho_d(A,B) = \rho_d(B,A)$.

Lemma 3 *Let A and B are inverse instances and π and π' – inverse schedules, then $L^A_{\max}(\pi^A) = L^B_{\max}(\pi^B)$.*

Lemma 4 *Let $A = \{G^A, (r^A_j, p_j, d^A_j) | \, j \in N\}$ and $B = \{G^B, (r^B_j, p_j, d^B_j) | \, j \in N\}$ (with identical processing times $p_j, j \in N$,) are two any instances then*

$$0 \leq L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \leq \rho_r(A,B). \tag{6}$$

Lemma 5 *Let $A = \{G, (r_j, p^A_j, d_j) | \, j \in N\}$ and $B = \{G, (r_j, p^B_j, d_j) | \, j \in N\}$ (with identical release times and due dates $r_j, d_j, j \in N$, and graph preceding G) are two any instances then*

$$0 \leq L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \leq \sum_{j\in N} |p^A_j - p^B_j| = \rho_r(A,B). \tag{7}$$

Theorem 1 *Let $A = \{G, (r^A_j, p^A_j, d^A_j) | \, j \in N\}$ and $B = \{G, (r^B_j, p^B_j, d^B_j) | \, j \in N\}$ (with identical graph preceding G) are two any instances then*

$$0 \leq L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \leq \rho(A,B). \tag{8}$$

¿From "symmetric" the instances A and B holds

$$0 \leq L^B_{max}(\pi^A) - L^B_{max}(\pi^B) \leq \rho(A,B) = \rho(B,A). \tag{9}$$

Theorem 2 *Let $A = \{G, (r^A_j, p^A_j, d^A_j) | \, j \in N\}$ and $B = \{G, (r^B_j, p^B_j, d^B_j) | \, j \in N\}$ (with identical graph preceding G) are two any instances then*

$$0 \leq L^A_{max}(\overline{\pi}) - L^A_{max}(\pi^A) \leq \delta^B(\overline{\pi}) + \rho(A,B), \tag{10}$$

where $\delta^B(\overline{\pi}) = L^B_{max}(\overline{\pi}) - L^B_{max}(\pi^B)$.

3 The scheme of approached decision of the problem

The idea of the approached decision of the problem consists of two stages. On the first step to the initial instance $A = \{G, (r_j^A, p_j^A, d_j^A) | j \in N\}$ is such change of its parameters r_j, p_j and d_j that the received instance $B = \{G, (r_j^B, p_j^B, d_j^B) | j \in N\}$ is belonged to a set polynomial solvable instances of the initial problem. On the next step we'll find optimal schedule to instance B. According to theorem 1 the schedule π^B to instance A have $0 \leq L_{max}^A(\pi^B) - L_{max}^A(\pi^A) \leq \rho(A, B)$.

Let's consider a case when class of polynomial solvable instances of the problem is defined by system of k linear inequalities

$$\mathbf{X} * \mathbf{R} + \mathbf{Y} * \mathbf{P} + \mathbf{Z} * \mathbf{D} \leq \mathbf{H}, \tag{11}$$

(s. t. $p_j \geq 0$, $\forall j \in N$), where $R = (r_1, \ldots, r_n)^T$, $P = (p_1, \ldots, p_n)^T$, $D = (d_1^C, \ldots, d_n)^T$, and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ – matrixes of dimension $k \times n$, and $H = (h_1, \ldots, h_k)^T$ – k-dimensional vector (the up index ^{T} designates transposition). Then in this class of instances (11) we find instance B with minimum "distance" $\rho(A, B)$ (to the initial instance A),

$$\begin{cases} \min\,(x^d - y^d + x^r - y^r) + \sum_{j \in N} x_j^p \\ y^d \leq d_j^A - d_j^B \leq x^d, \quad \forall j \in N, \\ y^r \leq r_j^A - r_j^B \leq x^r, \quad \forall j \in N, \\ -x_j^p \leq p_j^A - p_j^B \leq x_j^p, \quad \forall j \in N, \\ 0 \leq x_j^p, \quad \forall j \in N, \\ \mathbf{X} * \mathbf{R^B} + \mathbf{Y} * \mathbf{P^B} + \mathbf{Z} * \mathbf{D^B} \leq \mathbf{H}. \end{cases} \tag{12}$$

Linear programming problem (12) with $3n + 4 + n$ variables ($r_j^B, p_j^B, d_j^B, j = 1, \ldots, n$, and x^d, y^d, x^r, y^r, and x_j^p, $j = 1, \ldots, n$) and $7n + k$ inequalities can be sometimes solved in polynomial time considering specificity of linear restrictions. For example, for the problem $1|r_j|L_{max}$ two cases have been allocated polynomial solvable instances:

$$\begin{cases} d_1 \leq \ldots \leq d_n, \\ d_1 - r_1 - p_1 \geq \ldots \geq d_n - r_n - p_n \end{cases} \tag{13}$$

and

$$\max_{k \in N}\{d_k - r_k - p_k\} \leq d_j - r_j, \forall j \in N. \tag{14}$$

In the case (13) of the problem $1|r_j|L_{max}$ can be solved for polynomial time – $O(n^3 \log n)$ operations [16], [18]. And the task of linear programming (12) has been can be solved for polynomial time – $O(n \log n)$ operations [16], [17]. In the case (14) the problem can be solved in $O(n^2 \log n)$ operations [19]. As well as in case of (13) the minimum of absolute error of maximum lateness can be solved for polynomial time – in $O(n)$ operations [17].

If does not exist polynomial solvable instances or the "distance" $\rho(A, B)$ to any polynomial solvable point B is too "great", but we know for some instance $B = \{G, (r_j^B, p_j^B, d_j^B) | j \in N\}$ the absolute error of maximum lateness, then for initial instance $A = \{G, (r_j^A, p_j^A, d_j^A) | j \in N\}$ we can find approximation schedule $\overline{\pi}$ with the guaranteed absolute error of maximum lateness $0 \leq L_{max}^A(\overline{\pi}) - L_{max}^A(\pi^A) \leq \delta^B(\overline{\pi}) + \rho(A, B)$, according to theorem 2.

4 The scheme of approached decision for "close" problems

Let's need to find optimal schedule for some instance A to the problem $\alpha|\beta|L_{\max}$ and we know that the corresponding problem $\alpha|\beta|C_{\max}$ is polynomially solvable. Then all parameters except d_j

$(d_j^B = 0), \forall j \in N$, of instance B will be the analogous. Thus

$$0 \le L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \le \rho(A, B) = \max_{j\in N} d_j^A - \min_{j\in N} d_j^A.$$

Then let's need to find optimal schedule for instance A to the problem $\alpha|\beta|L_{\max}$ and we know that the corresponding problem $\alpha|\beta, p_j = p|L_{\max}$ is polynomially solvable. All parameters except $p_j, \forall j \in N$, of instance B will be the same. Thus we should decide optimization task

$$\rho(A, B) = \sum_j |p_j - p| \to \min_p .$$

Solution of the task is $p^* = p_{[\frac{n+1}{2}]}$ (if $p_1 \le \ldots \le p_n$). So we draw

$$L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \le \sum_{j=1}^n \left| p_j - p_{[\frac{n+1}{2}]} \right| .$$

If there is constrain $p_j = 1$, instead of $p_j = p, \forall j \in N$, then absolute error of maximum lateness to meet next condition

$$L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \le \sum_{j=1}^n \left| p_j - p_{[\frac{n+1}{2}]} \right| + 2p_{[\frac{n+1}{2}]}.$$

Let's our problem is $R|\beta|L_{\max}$ or $Q|\beta|L_{\max}$ then for the initial instance A of the problem we consider the problem $P|\beta|L_{\max}$ that is polynomially solvable. All parameters except $p_{ji}, \forall j \in N, \forall i \in M$, of instance B will be the same. Thus we should decide next optimization task

$$\rho(A, B) = \sum_{j\in N} \left(\max_{i\in M}\{(p_{ji}^A - p_j^B), 0\} - \min_{i\in M}\{(p_{ji}^A - p_j^B), 0\} \right) \to \min_{p_j} .$$

So we can take any value for p_j^B from interval $[\min_{i\in M} p_{ji}^A, \max_{i\in M} p_{ji}^A], \forall j \in N$, and

$$L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \le \sum_{j\in N} \left(\max_{i\in M} p_{ji}^A - \min_{i\in M} p_{ji}^A \right) .$$

Let's need to find optimal schedule for instance A to the $R|\beta|L_{\max}$ and we know that the corresponding problem $Q|\beta|L_{\max}$ is polynomially solvable. Then all parameters except $p_{ji}, \forall j \in N, \forall i \in M$, of instance B will be the same. In Q-problem $p_{ji} = p_j\sigma_i, \forall j \in N, \forall i \in M$. Thus we need to decide next optimization task

$$\rho(A, B) = \sum_{j\in N} \left(\max_{i\in M}\{(p_{ji}^A - p_j^B\sigma_i^B), 0\} - \min_{i\in M}\{(p_{ji}^A - p_j^B\sigma_i^B), 0\} \right) \to \min_{p_j^B, \sigma_i^B} .$$

The task can be represented as linear programming problem and be decided by simplex-method:

$$\begin{cases} \sum_{j\in N} (\alpha_j - \beta_j) \to \min_{\alpha_j, \beta_j, p_j^B, \sigma_i^B} \\ \beta_j \le p_{ji}^A - p_j^B\sigma_i^B \le \alpha_j, \qquad \forall j \in N, \forall i \in M, \\ \beta_j \le 0 \le \alpha_j, \qquad \forall j \in N. \end{cases}$$

The question of change the graph preceding is not considered yet...

References

[1] P. Brucker, M.R. Garey, D.S. Johnson (1977), Scheduling equal-length tasks under treelike precedence constraints to minimize maximum lateness, *Math. Oper. Res.* **2**(3), 275 – 284.

[2] J. Du, J.Y.-T. Leung, G.H. Young (1991), Scheduling chain-structured tasks to minimize makespan and mean flow time, *Inform. and Comput.* **92**(2), 219 – 236.

[3] M.R. Garey, D.S. Johnson (1978), "Strong" NP-completeness results: motivation, examples, and implications, *J. Assoc. Comput. Mach.* **25**(3), 499 – 508.

[4] J.K. Lenstra, A.H.G. Rinnooy Kan, P. Brucker (1977), Complexity of machine scheduling problems, Studies in integer programming (Proc. Workshop, Bonn, 1975), *Ann. of Discrete Math.* **1**, 343 – 362.

[5] J.D. Ullman (1975), NP-complete scheduling problems, *J. Comput. System Sci.* **10**, 384 – 393.

[6] M.R. Garey, D.S. Johnson (1977), Two-processor scheduling with start-times and deadlines, *SIAM J. Comput.*, **6**(3), 416 – 426.

[7] B. Simons (1983), Multiprocessor scheduling of unit-time jobs with arbitrary release times and deadlines, *SIAM J. Comput.*, **12**(2), 294 – 299.

[8] J. Blazewicz (1976), Scheduling dependent tasks with different arrival times to meet deadlines, *Model. Perform. Eval. Comput. Syst., Proc. int. Workshop, Stresa*, 57 – 65.

[9] P. Baptiste, P. Brucker, S. Knust, V. Timkovsky (2004), Ten notes on equal-execution-time scheduling, *Quarterly Journal of the Belgian, French and Italian Operations Research Societies* **2**, 111 – 127.

[10] K.R. Baker, E.L. Lawler, J.K. Lenstra, A.H.G Rinnooy Kan (1983), Preemptive scheduling of a single machine to minimize maximum cost subject to release dates and precedence constraints, *Oper. Res.* **31**, 381 – 386.

[11] M. Dror, W. Kubiak, P. Dell' Olmo (1998), Strong-weak chain constrained scheduling, *Ricerca Operativa* **27**, 35 – 49.

[12] M.R. Garey, D.S. Johnson (1976), Scheduling tasks with nonuniform deadlines on two processors, *J. Assoc. Comput. Mach.* **23**, 461 – 467.

[13] B. Jurisch (1995), Lower bounds for the job-shop scheduling problem on multi-purpose machines, *Discrete Appl. Math., Discrete Applied Mathematics. Combinatorial Algorithms, Optimization and Computer Science* **58**(2), 145 – 156.

[14] E.L. Lawler (1973), Optimal sequencing of a single machine subject to precedence constraints, *Management Sci.* **19**, 544 – 546.

[15] B. Simons (1978), A fast algorithm for single processor scheduling, *19th Annual Symposium on Foundations of Computer Science (Ann Arbor, Mich.)*, 246 – 252.

[16] A.A. Lazarev (1989), Efficient algorithms of decisions some problems of scheduling theory for single machine with due dates of service jobs, *PhD Dissertation*: 108 p (in Russian).

[17] A.A. Lazarev, R.R. Sadykov, S.V. Sevastianov (2006), The scheme of the approached decision of a single machine to minimize maximum lateness, *The discrete analysis and operations research* **2, 13, 1**: 57 – 76 (in Russian).

[18] A.A. Lazarev (2006), The Pareto-Optimal Set of the NP-Hard Problem of Minimizing of the Maximum Lateness for a Single Machine, *Journal of Computer and System Sciences International* **45**(6), 943 – 949.

[19] J.A. Hoogeveen (1996), Minimizing maximum promptness and maximum lateness on a single machine, *J. Math. Oper. Res.* **21**(1), 100 – 114.